

Precise deviations for Hawkes processes

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Abstract

Hawkes process is a class of simple point processes with self-exciting and clustering properties. Hawkes process has been widely applied in finance, neuroscience, social networks, criminology, seismology, and many other fields. In this paper, we study precise deviations for Hawkes processes for large time asymptotics, that strictly extends and improves the existing results in the literature.

1 Introduction

We consider the Hawkes process, a simple point process N_t , with the stochastic intensity at time t given by:

$$\lambda_t = \nu + \int_0^{t-} h(t-s) dN_s, \quad (1.1)$$

where $\nu > 0$ and $h : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ being locally bounded. We assume that $N_0 = 0$, that is, the Hawkes process starts at time zero with empty history. The Hawkes process is named after Alan Hawkes [22]. In the literature ν is called the baseline intensity, and h is called the exciting function, or kernel function, encoding the influence of past events on the intensity. Brémaud and Massoulié [9] generalized the dynamics (1.1) and the formula for the intensity (1.1) by a nonlinear function of $\int_0^{t-} h(t-s) dN_s$, and hence came the name nonlinear Hawkes processes. The original model (1.1) proposed by Hawkes [22] is thus sometimes referred to as the linear Hawkes process.

For the Hawkes process (1.1), the occurrence of a jump increases the intensity of the point process, and thus increases the likelihood of more future jumps. On the other hand, the intensity declines when there is no occurrence of new jumps. The self-exciting and clustering property makes the Hawkes process very appealing in applications in finance and many other fields. The Hawkes process is widely used in the modeling of the limit order books in high frequency trading, see e.g. Alfonsi and Blanc [4] for optimal execution, and Abergel and Jedidi [1] for ergodicity in Hawkes based limit order books models, and also the modeling of the duration between trades, see e.g. Bauwens and Hautsch [7] or the arrival process of buy and sell orders, see e.g. Bacry et al. [6]. The Hawkes process also finds applications in dark pool trading [18]. In the context of credit risk modeling,

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Errais et al. [13] used a top down approach using the affine point process, which includes Hawkes process as a special case. Giot [19], Chavez-Demoulin et al. [10] tested Hawkes processes in the risk management context. Aït-Sahalia et al. [2, 3] used Hawkes processes to model two key aspects of asset prices: clustering in time and cross sectional contamination between regions. The Hawkes process has also been used to explain the supply and demand microstructure in an interest rate model in Hainaut [21]. The applications other than finance include: neuroscience, see e.g. [32, 33, 35, 36], genome analysis, see e.g. [20, 36], networks and sociology, see e.g. [11, 28, 41], queueing theory, see e.g. [17], insurance, see e.g. [37, 42], criminology, see e.g. [27, 29, 34], seismology, see e.g. [30, 31, 40] and many other fields.

In this paper, we consider the linear Hawkes process N_t with $N_0 = 0$ with the intensity (1.1). We assume throughout this paper that

- $\|h\|_{L^1} = \int_0^\infty h(t)dt < 1$.
- $\int_0^\infty th(t)dt < \infty$.

Let us first review the limit theorems for linear Hawkes processes in the literature. It is well known that under the assumption $\|h\|_{L^1} < 1$, there exists a unique stationary Hawkes process, and we have the law of large numbers $\frac{N_t}{t} \rightarrow \frac{\nu}{1-\|h\|_{L^1}}$ as $t \rightarrow \infty$. Bacry et al. [5] obtained a functional central limit theorem for multivariate Hawkes process and as a special case of their result, we have

$$\frac{N_t - \frac{\nu}{1-\|h\|_{L^1}}t}{\sqrt{t}} \rightarrow N\left(0, \frac{\nu}{(1-\|h\|_{L^1})^3}\right), \quad (1.2)$$

in distribution as $t \rightarrow \infty$ under the assumption that $\int_0^\infty t^{1/2}h(t)dt < \infty$. Bordenave and Torrisi [8] proved that $\mathbb{P}(\frac{N_t}{t} \in \cdot)$ satisfies a large deviation principle, with the rate function:

$$I(x) = \begin{cases} x \log\left(\frac{x}{\nu+x\|h\|_{L^1}}\right) - x + x\|h\|_{L^1} + \nu & \text{if } x \in [0, \infty) \\ +\infty & \text{otherwise} \end{cases}. \quad (1.3)$$

Note that the rate function in the paper [8] is written as the Legendre transform expression, and the formula (1.3) is first mentioned in [44]. Also notice that in [8], the assumption $\int_0^\infty th(t)dt < \infty$ is needed, and indeed this assumption is not necessary, see e.g. [26]. A moderate deviation principle is obtained in [44] that fills in the gap between the central limit theorem and the large deviation principle. Other works on the asymptotics of linear Hawke processes, including the nearly unstable Hawkes processes, that is, when the $\|h\|_{L^1}$ is close to 1, see e.g. [24, 25], and the large initial intensity asymptotics for the Markovian case [15, 16], and the large baseline intensity asymptotics [17].

For nonlinear Hawkes processes, [43] studies the central limit theorem, and [45] obtains a process-level, i.e., level-3 large deviation principle, and hence has the scalar large deviations as a by-product. An alternative expression for the rate function when the system

is Markovian is obtained in [46]. Recently, Torrisi [38, 39] studies the rate of convergence in the Gaussian and Poisson approximations of the simple point processes with stochastic intensity, which includes as a special case, the nonlinear Hawkes process.

The large deviations [8] and moderate deviations [44] for linear Hawkes processes are of the Donsker-Varadhan type, which only gives the leading order term, but not the higher order expansion. In many applications in finance, insurance, and other fields, more precise deviations are desired, which motivates us to study the precise deviations for linear Hawkes processes. In this paper, we will derive the precise deviations for linear Hawkes processes, using the recent mod- ϕ convergence theory developed in [14]. The moment generating function for linear Hawkes processes has semi-explicit formula due to the immigration-birth representation of linear Hawkes processes, and then the precise deviations results follow from the mod- ϕ convergence theory after careful analysis and a series of propositions and lemmas. The paper is organized as follows. We will state the main results of our paper in Section 2. In particular, we will give precise large deviations results in Section 2.1 and precise moderate deviations and fluctuation results in Section 2.2. All the proofs will be provided in Section 3.

2 Main Results

In this paper, we apply the recently developed mod- ϕ convergence method to obtain precise large deviations for linear Hawkes processes for the large time asymptotic regime. We will also obtain the precise moderate deviations and some fluctuations results.

Let us first recall the definition of mod- ϕ convergence, see e.g. Definition 1.1. [14]. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of real-valued random variables and $\mathbb{E}[e^{zX_n}]$ exist in a strip $\mathcal{S}_{(c,d)} := \{z \in \mathbb{C} : c < \mathcal{R}(z) < d\}$, with $c < d$ extended real numbers, i.e. we allow $c = -\infty$ and $d = +\infty$ and $\mathcal{R}(z)$ denotes the real part of $z \in \mathbb{C}$ throughout this paper. We assume that there exists a non-constant infinitely divisible distribution ϕ with $\int_{\mathbb{R}} e^{zx} \phi(dx) = e^{\eta(z)}$, which is well defined on $\mathcal{S}_{(c,d)}$, and an analytic function $\psi(z)$ that does not vanish on the real part of $\mathcal{S}_{(c,d)}$ such that locally uniformly in $z \in \mathcal{S}_{(c,d)}$,

$$e^{-t_n \eta(z)} \mathbb{E}[e^{zX_n}] \rightarrow \psi(z), \quad (2.1)$$

where $t_n \rightarrow +\infty$ as $n \rightarrow \infty$. Then we say that X_n converges mod- ϕ on $\mathcal{S}_{(c,d)}$ with parameters $(t_n)_{n \in \mathbb{N}}$ and limiting function ψ .

Assume that ϕ is a lattice distribution. Then Theorem 3.4. [14] says that for any $x \in \mathbb{R}$ in the interval $(\eta'(c), \eta'(d))$ and θ^* defined as $\eta'(\theta^*) = x$, assume that $t_n x \in \mathbb{N}$, then,

$$\mathbb{P}(X_n = t_n x) = \frac{e^{-t_n F(x)}}{\sqrt{2\pi t_n \eta''(\theta^*)}} \left(\psi(\theta^*) + \frac{a_1}{t_n} + \frac{a_2}{t_n^2} + \cdots + \frac{a_{v-1}}{t_n^{v-1}} + O\left(\frac{1}{t_n^v}\right) \right), \quad (2.2)$$

as $n \rightarrow \infty$, where $F(x) := \sup_{\theta \in \mathbb{R}} \{\theta x - \eta(\theta)\}$ is the Legendre transform of $\eta(\cdot)$, and

similarly, if $x \in \mathbb{R}$ is in the range of $(\eta'(0), \eta'(d))$, then,

$$\mathbb{P}(X_n \geq t_n x) = \frac{e^{-t_n F(x)}}{\sqrt{2\pi t_n \eta''(\theta^*)}} \frac{1}{1 - e^{-\theta^*}} \left(\psi(\theta^*) + \frac{b_1}{t_n} + \frac{b_2}{t_n^2} + \cdots + \frac{b_{v-1}}{t_n^{v-1}} + O\left(\frac{1}{t_n^v}\right) \right), \quad (2.3)$$

as $n \rightarrow \infty$, where $(a_k)_{k=1}^\infty$, $(b_k)_{k=1}^\infty$ are rational fractions in the derivatives of η and ψ at θ^* , that can be computed as described in Remark 3.7. [14].

2.1 Precise Large Deviations

Our main results for the precise large deviations for the Hawkes process is stated as follows. It provides the full expansion to arbitrary order in the large time asymptotic regime, which generalizes and the large deviations result in [8].

Theorem 1. (i) For any $x > 0$, and $tx \in \mathbb{N}$, as $t \rightarrow \infty$,

$$\mathbb{P}(N_t = tx) = e^{-tI(x)} \sqrt{\frac{I''(x)}{2\pi t}} \left(\psi(\theta^*) + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_{v-1}}{t^{v-1}} + O\left(\frac{1}{t^v}\right) \right), \quad (2.4)$$

where for any $\mathcal{R}(z) \leq \|h\|_{L^1} - 1 - \log \|h\|_{L^1}$,

$$\psi(z) := e^{\nu\varphi(z)}, \quad \text{and} \quad \varphi(z) := \int_0^\infty [F(s; z) - x(z)] ds, \quad (2.5)$$

which is analytic in z for any $\mathcal{R}(z) < \|h\|_{L^1} - 1 - \log \|h\|_{L^1}$, and F is the unique solution that satisfies

$$F(t; z) = e^{z + \int_0^t (F(t-s; z) - 1) h(s) ds}, \quad (2.6)$$

with the constraint $|F(t; z)| \leq \frac{1}{\|h\|_{L^1}}$, and it is analytic in z for any $\mathcal{R}(z) < \|h\|_{L^1} - 1 - \log \|h\|_{L^1}$, and $x(z) := F(\infty; z)$ exists and it satisfies the equation

$$x(z) = e^{z + \|h\|_{L^1}(x(z) - 1)}, \quad (2.7)$$

and it is analytic in z for any $\mathcal{R}(z) < \|h\|_{L^1} - 1 - \log \|h\|_{L^1}$. And $I(x)$ is defined in (1.3), $I''(x) = \frac{\nu^2}{x(\nu + \|h\|_{L^1} x)^2}$, and

$$\theta^* = \log \left(\frac{x}{\nu + \|h\|_{L^1} x} \right) - \frac{\|h\|_{L^1} x}{\nu + \|h\|_{L^1} x} + \|h\|_{L^1}, \quad (2.8)$$

where $(a_k)_{k=1}^\infty$ are rational fractions in the derivatives of η and ψ at θ^* , where $\eta(z) := \nu(x(z) - 1)$.

(ii) For any $x > \frac{\nu}{1 - \|h\|_{L^1}}$, as $t \rightarrow \infty$,

$$\mathbb{P}(N_t \geq tx) = e^{-tI(x)} \sqrt{\frac{I''(x)}{2\pi t}} \frac{1}{1 - e^{-\theta^*}} \left(\psi(\theta^*) + \frac{b_1}{t} + \frac{b_2}{t^2} + \cdots + \frac{b_{v-1}}{t^{v-1}} + O\left(\frac{1}{t^v}\right) \right), \quad (2.9)$$

where $(b_k)_{k=1}^\infty$ are rational fractions in the derivatives of η and ψ at θ^* .

The main strategy of the proof of Theorem 1 is by showing the mod- ϕ convergence as defined in [14] and apply their Theorem 3.4. The proof of Theorem 1 relies on a series of lemmas and propositions that we will state later. We will first recall and discuss some well-known properties for the linear Hawkes process that will be used extensively in our proofs later.

Hawkes and Oakes [23] first discovered that a linear Hawkes process has an immigration-birth representation. The immigrants (roots) arrive according to a standard Poisson process \tilde{N} with intensity $\nu > 0$ at time t . Each immigrant generates children according to a Galton-Watson tree, that is, the number of children of each immigrant follows a Poisson distribution with parameter $\|h\|_{L^1}$, and each child will independently generate children according to the same Poisson distribution, and so on and so forth. In addition, when the children are born, they are born at the same time, with the probability density function $\frac{h(t)}{\|h\|_{L^1}}$ for being born at time t . Then, the Hawkes process N_t is the number of all the immigrants and their descendants that arrive on the time interval $[0, t]$.

By the immigration-birth representation for linear Hawkes processes, it is well-known that one can compute that, see e.g. [44, 26, 18], for any $z \in \mathbb{C}$, such that

$$\mathcal{R}(z) < \theta_c := \|h\|_{L^1} - 1 - \log \|h\|_{L^1}, \quad (2.10)$$

we have

$$\mathbb{E}[e^{zN_t}] = e^{\nu \int_0^t (F(s; z) - 1) ds}, \quad (2.11)$$

where F satisfies the equation:

$$F(t; z) = e^{z + \int_0^t (F(t-s; z) - 1) h(s) ds}, \quad (2.12)$$

for any $t \geq 0$.

Note that by the immigration-birth representation, we can interpret $F(t; z)$ as:

$$F(t; z) = \mathbb{E}[e^{zS_t}], \quad (2.13)$$

where S_t is the number of all the descendants of an immigrant that arrives at time 0, on the time interval $[0, t]$ including the immigrant. Moreover, let us define:

$$x(z) = \mathbb{E}[e^{zS_\infty}]. \quad (2.14)$$

It is well known that $x(z)$ satisfies the algebraic equation, see e.g. [26]:

$$x(z) = e^{z + \|h\|_{L^1}(x(z) - 1)}. \quad (2.15)$$

This algebraic equation may have more than one solution. It is known that for $z \in \mathbb{R}$, there are at most two solutions of this algebraic equation and $\mathbb{E}[e^{zS_\infty}]$ is the smaller solution, see e.g. [26].

By dominated convergence theorem, for $\mathcal{R}(z) < \theta_c$, where θ_c is defined in (2.10),

$$F(t, z) \rightarrow x(z), \quad \text{as } t \rightarrow \infty. \quad (2.16)$$

The limit $x(\cdot)$ has the following properties:

Proposition 2. *For any $\theta \in \mathbb{R}$, and $\theta \leq \theta_c$, where θ_c is defined in (2.10), we have*

- (i) $x(\theta)\|h\|_{L^1} \leq 1$.
- (ii) $x'(\theta) \rightarrow \infty$ as $\theta \uparrow \theta_c$.

We know that $\mathbb{E}[e^{zS_t}]$ satisfies the equation (2.6). Thus, as a by-product of the immigration-birth representation for linear Hawkes processes, we get the existence of the solution of (2.6).

Let us notice that for any $\mathcal{R}(z) \leq \theta_c$,

$$|F(t; z)| = |\mathbb{E}[e^{zS_t}]| \leq \mathbb{E}[|e^{zS_t}|] = \mathbb{E}[e^{\mathcal{R}(z)S_t}] \leq \frac{1}{\|h\|_{L^1}}. \quad (2.17)$$

Therefore, it suffices to consider the solution of the equation (2.6) that satisfies the constraint $|F(t; z)| \leq \frac{1}{\|h\|_{L^1}}$.

With this additional constraint, the equation (2.6) has a unique solution:

Proposition 3. *Let $z \in \mathbb{C}$ and $\mathcal{R}(z) \leq \theta_c$. The equation (2.6) with the constraint $|F(t; z)| \leq \frac{1}{\|h\|_{L^1}}$ has a unique solution.*

The key to prove the main result Theorem 1 is to verify the mod- ϕ convergence. More precisely, we need to show that

$$e^{-t\eta(z)}\mathbb{E}[e^{zN_t}] \rightarrow \psi(z) := e^{\nu\varphi(z)}, \quad (2.18)$$

as $t \rightarrow \infty$ locally uniformly in z for $\mathcal{R}(z) < \|h\|_{L^1} - 1 - \log \|h\|_{L^1}$, where

$$\varphi(z) = \int_0^\infty [F(s; z) - x(z)]ds, \quad (2.19)$$

is analytic in z and

$$\eta(z) = \nu(x(z) - 1), \quad (2.20)$$

and

$$e^{\eta(z)} = e^{\nu(x(z)-1)} = \mathbb{E}[e^{zY}], \quad (2.21)$$

for some random variable Y , where Y has an infinitely divisible distribution.

We will show the mod- ϕ convergence below via a series of lemmas.

First, we show that Y is infinitely divisible. The infinite divisibility is a limitation of the method of mod- ϕ convergence. Fortunately, the limiting distribution in the case of the linear Hawke process is indeed infinitely divisible.

Lemma 4. *Y has an infinitely divisible distribution.*

To show the mod- ϕ convergence, the main technical lemma is given as follows:

Lemma 5. For any $\mathcal{R}(z) < \theta_c$, where θ_c is defined in (2.10),

$$\varphi(z) = \int_0^\infty [F(s; z) - x(z)] ds \quad (2.22)$$

is well-defined and analytic, and as $t \rightarrow \infty$,

$$e^{-t(\nu(x(z)-1))} \mathbb{E}[e^{zN_t}] \rightarrow e^{\nu\varphi(z)}, \quad (2.23)$$

locally uniformly in z .

To this end, we have established the mod- ϕ convergence for the linear Hawkes process for the large time limit. The proofs of all the propositions, lemmas and Theorem 2.1 will be given in Section 3.

2.2 Precise Moderate Deviations and Fluctuations

The mod ϕ convergence implies also the precise moderate deviations and central limit theorem, see Theorem 3.9. [14].

By Theorem 3.9. [14], we have the following central limit theorem result:

Theorem 6. For any $y = o(t^{1/6})$, as $t \rightarrow \infty$,

$$\mathbb{P}\left(N_t \geq \frac{\nu}{1 - \|h\|_{L^1}} t + \sqrt{t} \frac{\sqrt{\nu}}{(1 - \|h\|_{L^1})^{3/2}} y\right) = \bar{\Phi}(y)(1 + o(1)), \quad (2.24)$$

where $\bar{\Phi}(y) := \int_y^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$.

Remark 7. Note that Theorem 6 generalizes the univariate case of the Hawkes process central limit theorem considered in [5] since here we allow $y = o(t^{1/6})$ for $t \rightarrow \infty$.

By Theorem 3.9. [14], we can also study the moderate deviations result. If $1 \ll y \ll \sqrt{t}$ for $t \rightarrow \infty$, i.e., the moderate deviations regime, and if we let:

$$s_t := \frac{\nu}{1 - \|h\|_{L^1}} t + \sqrt{t} \frac{\sqrt{\nu}}{(1 - \|h\|_{L^1})^{3/2}} y, \quad (2.25)$$

then, as $t \rightarrow \infty$,

$$\mathbb{P}\left(N_t \geq \frac{\nu}{1 - \|h\|_{L^1}} t + \sqrt{t} \frac{\sqrt{\nu}}{(1 - \|h\|_{L^1})^{3/2}} y\right) = \frac{e^{-tI(s_t)}}{\theta^* \sqrt{2\pi t \eta''(\theta^*)}} (1 + o(1)), \quad (2.26)$$

where $\eta'(\theta^*) = s_t$.

Corollary 3.13. [14] gives a more explicit form of Theorem 3.9. [14]. By using Corollary 3.13. [14], we have the following result:

Theorem 8. (i) If $y = o(t^{1/4})$, then as $t \rightarrow \infty$,

$$\mathbb{P} \left(N_t \geq \frac{\nu}{1 - \|h\|_{L^1}} t + \sqrt{t} \frac{\sqrt{\nu}}{(1 - \|h\|_{L^1})^{3/2}} y \right) = \frac{(1 + o(1))}{y\sqrt{2\pi}} e^{-\frac{y^2}{2}} e^{\frac{\eta'''(0)}{6(\eta''(0))^{3/2}} \frac{y^3}{\sqrt{t}}}. \quad (2.27)$$

(ii) If $y = o(t^{1/2-1/m})$, where $m \geq 3$, then as $t \rightarrow \infty$,

$$\mathbb{P} \left(N_t \geq \frac{\nu}{1 - \|h\|_{L^1}} t + \sqrt{t} \frac{\sqrt{\nu}}{(1 - \|h\|_{L^1})^{3/2}} y \right) = \frac{(1 + o(1))}{y\sqrt{2\pi}} e^{-\sum_{i=2}^{m-1} \frac{I^{(i)}(\eta'(0))}{i!} \frac{(\eta''(0))^{i/2} y^i}{t^{(i-2)/2}}}, \quad (2.28)$$

where $I(\cdot)$ is defined in (1.3), and

$$\eta'(0) = \frac{\nu}{1 - \|h\|_{L^1}}, \quad \eta''(0) = \frac{\nu}{(1 - \|h\|_{L^1})^3}, \quad \eta'''(0) = \nu \frac{1 + 2\|h\|_{L^1}}{(1 - \|h\|_{L^1})^5}. \quad (2.29)$$

Remark 9. Note that Theorem 8 (i) gives a precise moderate deviation result, and it provides a more precise tail estimate than [44]. To see this, let $y = \frac{(1 - \|h\|_{L^1})^{3/2}}{\sqrt{\nu}} \frac{a(t)}{\sqrt{t}} x$, where x is a constant independent of t . Then for $t^{1/2} \ll a(t) \ll t^{3/4}$, as $t \rightarrow \infty$, we have

$$\begin{aligned} & \mathbb{P} \left(N_t \geq \frac{\nu}{1 - \|h\|_{L^1}} t + a(t)x \right) \\ &= \frac{\sqrt{\nu}(1 + o(1))}{(1 - \|h\|_{L^1})^{3/2} \sqrt{2\pi}} \frac{\sqrt{t}}{a(t)x} e^{-\frac{(1 - \|h\|_{L^1})^3}{2\nu} \frac{a(t)^2}{t} x^2} e^{\frac{\eta'''(0)}{6(\eta''(0))^{3/2}} \frac{a(t)^3}{t^2} x^3} \\ &= \frac{\sqrt{\nu}(1 + o(1))}{(1 - \|h\|_{L^1})^{3/2} \sqrt{2\pi}} \frac{\sqrt{t}}{a(t)x} e^{-\frac{(1 - \|h\|_{L^1})^3}{2\nu} \frac{a(t)^2}{t} x^2} e^{\frac{(1+2\|h\|_{L^1})(1 - \|h\|_{L^1})^4}{6\nu^2} \frac{a(t)^3}{t^2} x^3}, \end{aligned}$$

where $\eta'(0)$, $\eta''(0)$, and $\eta'''(0)$ are given in (2.29).

3 Proofs

3.1 Proofs of the Results in Section 2.1

Proof of Proposition 2. Let us prove this first. Consider $\theta \in \mathbb{R}$, and

$$x(\theta) = e^{\theta + \|h\|_{L^1}(x(\theta) - 1)}. \quad (3.1)$$

Then $x(\theta)$ is increasing in θ by $x(\theta) = \mathbb{E}[e^{\theta Y}]$ and the definition of Y . Moreover, for $\theta = \theta_c = \|h\|_{L^1} - 1 - \log \|h\|_{L^1}$, we have

$$x(\theta_c) = \frac{1}{\|h\|_{L^1}} e^{\|h\|_{L^1} - 1 + \|h\|_{L^1}(x(\theta_c) - 1)}, \quad (3.2)$$

which implies that $x(\theta_c)\|h\|_{L^1} = 1$. Thus, for any $\theta \leq \theta_c$, $x(\theta)\|h\|_{L^1} \leq 1$.

We also notice that

$$x'(\theta) = (1 + \|h\|_{L^1} x'(\theta)) x(\theta), \quad (3.3)$$

and therefore

$$x'(\theta) = \frac{x(\theta)}{1 - \|h\|_{L^1} x(\theta)} \rightarrow \infty, \quad (3.4)$$

as $\theta \uparrow \theta_c$. \square

Proof of Proposition 3. Suppose F_1 and F_2 are two solutions that satisfy (2.6) with $|F_j| \leq \frac{1}{\|h\|_{L^1}}$, for $j = 1, 2$. Note that for any $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned} |e^{z_1} - e^{z_2}| &= \left| \sum_{n=1}^{\infty} \frac{z_1^n - z_2^n}{n!} \right| \\ &\leq |z_1 - z_2| \sum_{n=1}^{\infty} \frac{|z_1|^{n-1} |z_2| + \cdots + |z_1| |z_2|^{n-1}}{n!} \\ &\leq |z_1 - z_2| \sum_{n=1}^{\infty} \frac{n(|z_1| + |z_2|)^{n-1}}{n!} = |z_1 - z_2| e^{|z_1| + |z_2|}. \end{aligned}$$

Hence, for any $T > 0$ and for any $0 \leq t \leq T$,

$$\begin{aligned} |F_1(t; z) - F_2(t; z)| &= |e^z| \left| e^{\int_0^t (F_1(t-s; z) - 1) h(s) ds} - e^{\int_0^t (F_2(t-s; z) - 1) h(s) ds} \right| \\ &\leq |e^z| \left| \int_0^t (F_1(t-s; z) - F_2(t-s; z)) h(s) ds \right| \\ &\quad \cdot e^{|\int_0^t (F_1(t-s; z) - 1) h(s) ds| + |\int_0^t (F_2(t-s; z) - 1) h(s) ds|} \\ &\leq |e^z| \|h\|_{L^\infty[0, T]} e^{2\left(\frac{1}{\|h\|_{L^1}} + 1\right) \|h\|_{L^1}} \int_0^t |F_1(s; z) - F_2(s; z)| ds. \end{aligned}$$

Note that $F_1(0; z) = F_2(0; z) = e^z$ from (2.6). By Gronwall's inequality, we conclude that $F_1 \equiv F_2$. \square

Proof of Lemma 4. where Y can be interpreted as $Y = \sum_{i=1}^K Z_i$, where K is Poisson random variable with parameter ν and Z_i are i.i.d. random variable interpreted as the total number of the nodes in a Galton-Watson tree with the number of children being born in each generation Poisson distributed with parameter $\|h\|_{L^1}$. Then, it is clear that we can write $Y = \sum_{j=1}^n Y_j$ in distribution, where Y_j are i.i.d. $Y_j = \sum_{i=1}^{K_j} Z_{ij}$, where Z_{ij} are i.i.d. copies of Z_1 and K_j are i.i.d. Poisson distributed with parameter $\frac{\nu}{n}$. Therefore, Y has an infinitely divisible distribution. \square

Proof of Lemma 5. Firstly, it is obvious that for any $s > 0$, and $t > 0$,

$$F(s, z), \quad x(z), \quad \int_0^t (F(s, z) - x(z))ds,$$

are analytic in z for $\mathcal{R}(z) < \theta_c$.

Since $\|h\|_{L^1} < 1$ and $x(0) = 1$, for $\mathcal{R}(z) < \|h\|_{L^1} - 1 - \log \|h\|_{L^1}$, we have $|x(z)|\|h\|_{L^1} \leq x(\mathcal{R}(z))\|h\|_{L^1} < 1$. Therefore, for any compact subset $\mathbb{K} \subset \{z \in \mathbb{C}; \mathcal{R}(z) < \|h\|_{L^1} - 1 - \log \|h\|_{L^1}\}$, we have

$$\sup_{z \in \mathbb{K}} |x(z)|\|h\|_{L^1} \leq x(\mathcal{R}(z))\|h\|_{L^1} < 1.$$

Since $\int_s^\infty h(u)du \rightarrow 0$ as $s \rightarrow \infty$ and $F(s; z) = \mathbb{E}[e^{zS_s}] \rightarrow x(z) = \mathbb{E}[e^{zS_\infty}]$, we have that

$$\int_0^s \sup_{z \in \mathbb{K}} [F(s-u; z) - x(z)]h(u)du \rightarrow 0, \quad \text{as } s \rightarrow \infty.$$

Note that

$$F(s; z) - x(z) = x(z) \left[e^{\int_0^s [F(s-u; z) - x(z)]h(u)du - (x(z)-1) \int_s^\infty h(u)du} - 1 \right],$$

and for any fixed $\delta > 0$ such that $(1 + \delta) \sup_{z \in \mathbb{K}} |x(z)|\|h\|_{L^1} < 1$, there exists $M > 0$, so that for any $s \geq M$ and $z \in \mathbb{K}$, we have

$$|F(s; z) - x(z)| \leq (1 + \delta)|x(z)| \left[\int_0^s |F(s-u; z) - x(z)|h(u)du + |x(z) - 1| \int_s^\infty h(u)du \right].$$

Therefore, we get that for any $T > M$,

$$\begin{aligned} \int_M^T \sup_{z \in \mathbb{K}} |F(s; z) - x(z)|ds &\leq (1 + \delta) \sup_{z \in \mathbb{K}} |x(z)| \int_M^T \int_0^s \sup_{z \in \mathbb{K}} |F(s-u; z) - x(z)|h(u)duds \\ &\quad + (1 + \delta) \sup_{z \in \mathbb{K}} |x(z)||x(z) - 1| \int_M^T \int_s^\infty h(u)duds \\ &\leq (1 + \delta) \sup_{z \in \mathbb{K}} |x(z)| \int_0^T \int_0^s \sup_{z \in \mathbb{K}} |F(s-u; z) - x(z)|h(u)duds \\ &\quad + (1 + \delta) \sup_{z \in \mathbb{K}} |x(z)||x(z) - 1| \int_0^\infty \int_s^\infty h(u)duds, \end{aligned}$$

which implies that

$$\begin{aligned}
& \int_0^T \sup_{z \in \mathbb{K}} |F(s; z) - x(z)| ds \\
& \leq \int_0^M \sup_{z \in \mathbb{K}} |F(s; z) - x(z)| ds + (1 + \delta) \sup_{z \in \mathbb{K}} |x(z)| \int_0^T \int_0^s \sup_{z \in \mathbb{K}} |F(s - u; z) - x(z)| h(u) du ds \\
& \quad + (1 + \delta) \sup_{z \in \mathbb{K}} |x(z)| |x(z) - 1| \int_0^\infty \int_s^\infty h(u) du ds \\
& = \int_0^M \sup_{z \in \mathbb{K}} |F(s; z) - x(z)| ds + (1 + \delta) \sup_{z \in \mathbb{K}} |x(z)| \int_0^T \left[\int_u^T \sup_{z \in \mathbb{K}} |F(s - u; z) - x(z)| ds \right] h(u) du \\
& \quad + (1 + \delta) \sup_{z \in \mathbb{K}} |x(z)| |x(z) - 1| \int_0^\infty \int_s^\infty h(u) du ds \\
& \leq \int_0^M \sup_{z \in \mathbb{K}} |F(s; z) - x(z)| ds + (1 + \delta) \sup_{z \in \mathbb{K}} |x(z)| \|h\|_{L^1} \int_0^T |F(s; z) - x(z)| ds \\
& \quad + (1 + \delta) \sup_{z \in \mathbb{K}} |x(z)| \left[\sup_{z \in \mathbb{K}} |x(z)| + 1 \right] \int_0^\infty sh(s) ds,
\end{aligned}$$

which holds for any $T > M$, and thus we have

$$\begin{aligned}
& \int_0^\infty \sup_{z \in \mathbb{K}} |F(s; z) - x(z)| ds \\
& \leq \frac{\int_0^M \sup_{z \in \mathbb{K}} |F(s; z) - x(z)| ds + (1 + \delta) \sup_{z \in \mathbb{K}} |x(z)| [\sup_{z \in \mathbb{K}} |x(z)| + 1] \int_0^\infty sh(s) ds}{1 - (1 + \delta) \sup_{z \in \mathbb{K}} |x(z)| \|h\|_{L^1}}.
\end{aligned} \tag{3.5}$$

Hence, we conclude that $\int_t^\infty \sup_{z \in \mathbb{K}} |F(s; z) - x(z)| ds \rightarrow 0$ as $t \rightarrow \infty$, and so

$$\int_0^t (F(s, z) - x(z)) ds \rightarrow \int_0^\infty (F(s, z) - x(z)) ds,$$

as $t \rightarrow \infty$, locally uniformly in z for $\mathcal{R}(z) < \theta_c$. Hence, $\varphi(z)$ is well-defined and is analytic in z for $\mathcal{R}(z) < \theta_c$.

By equation (2.11), we have proved that locally uniformly in z for $\mathcal{R}(z) < \theta_c$,

$$e^{-t(\nu(x(z)-1))} \mathbb{E}[e^{zN_t}] = e^{\nu \int_0^t (F(s, z) - x(z)) ds} \rightarrow e^{\nu \varphi(z)}, \quad \text{as } t \rightarrow \infty.$$

□

Proof of Theorem 1. By Lemma 4, and Lemma 5, we have established the mod- ϕ convergence. Hence, by Theorem 3.4. [14], for any $x > 0$, and $tx \in \mathbb{N}$,

$$\mathbb{P}(N_t = tx) = \frac{e^{-tI(x)}}{\sqrt{2\pi t\eta''(\theta^*)}} \left(\psi(\theta^*) + \frac{a_1}{t} + \frac{a_2}{t^2} + \cdots + \frac{a_{v-1}}{t^{v-1}} + O\left(\frac{1}{t^v}\right) \right), \tag{3.6}$$

and for any $x > \eta'(0) = \frac{\nu}{1-\|h\|_{L^1}}$,

$$\mathbb{P}(N_t \geq tx) = \frac{e^{-tI(x)}}{\sqrt{2\pi t\eta''(\theta^*)}} \frac{1}{1-e^{-\theta^*}} \left(\psi(\theta^*) + \frac{b_1}{t} + \frac{b_2}{t^2} + \cdots + \frac{b_{v-1}}{t^{v-1}} + O\left(\frac{1}{t^v}\right) \right). \quad (3.7)$$

where $I(x)$ is defined in (1.3) and $\eta'(\theta^*) = x$.

Note that $\eta(\theta) = \nu(x(\theta) - 1)$, where $x(\theta) = e^{\theta + \|h\|_{L^1}(x(\theta)-1)}$. Thus, $\eta'(\theta) = \nu x'(\theta)$, and

$$x'(\theta) = (1 + \|h\|_{L^1} x'(\theta)) e^{\theta + \|h\|_{L^1}(x(\theta)-1)} = (1 + \|h\|_{L^1} x'(\theta)) x(\theta), \quad (3.8)$$

which implies that

$$x(\theta) = \frac{x'(\theta)}{1 + \|h\|_{L^1} x'(\theta)}. \quad (3.9)$$

Notice that $\eta'(\theta^*) = x$, and thus

$$x'(\theta^*) = \frac{\eta'(\theta^*)}{\nu} = \frac{x}{\nu}, \quad (3.10)$$

and

$$x(\theta^*) = \frac{x'(\theta^*)}{1 + \|h\|_{L^1} x'(\theta^*)} = \frac{x}{\nu + \|h\|_{L^1} x}. \quad (3.11)$$

Hence,

$$\begin{aligned} \frac{x}{\nu} &= x'(\theta^*) = (1 + \|h\|_{L^1} x'(\theta^*)) e^{\theta^* + \|h\|_{L^1}(x(\theta^*)-1)} \\ &= \left(1 + \frac{\|h\|_{L^1} x}{\nu}\right) e^{\theta^* + \frac{\|h\|_{L^1} x}{\nu + \|h\|_{L^1} x} - \|h\|_{L^1}}, \end{aligned}$$

which implies that

$$\theta^* = \log \left(\frac{x}{\nu + \|h\|_{L^1} x} \right) - \frac{\|h\|_{L^1} x}{\nu + \|h\|_{L^1} x} + \|h\|_{L^1}. \quad (3.12)$$

Hence, we have verified that

$$I(x) = \theta^* x - \eta(\theta^*) = x \log \left(\frac{x}{\nu + \|h\|_{L^1} x} \right) - x + x \|h\|_{L^1} + \nu. \quad (3.13)$$

Moreover, by the property of Legendre transform,

$$\eta''(\theta^*) = \frac{1}{I''(x)}, \quad (3.14)$$

and

$$I'(x) = \theta^* = \log \left(\frac{x}{\nu + \|h\|_{L^1} x} \right) - \frac{\|h\|_{L^1} x}{\nu + \|h\|_{L^1} x} + \|h\|_{L^1}, \quad (3.15)$$

which implies that

$$I''(x) = \frac{\nu^2}{x(\nu + \|h\|_{L^1} x)^2}. \quad (3.16)$$

□

3.2 Proofs of the Results in Section 2.2

Proof of Theorem 8. Since we have established mod- ϕ convergence, and N_t is lattice distributed, the result follows from Corollary 3.13. [14].

Let us show that (2.29) holds. Let us recall that $\eta(\theta) = \nu(x(\theta) - 1)$, where $x(\theta) = e^{\theta + \|h\|_{L^1}(x(\theta) - 1)}$. Thus, we can compute that

$$x'(\theta) = (1 + \|h\|_{L^1} x'(\theta)) x(\theta), \quad (3.17)$$

and

$$x''(\theta) = [\|h\|_{L^1} x''(\theta) + (1 + \|h\|_{L^1} x'(\theta))^2] x(\theta), \quad (3.18)$$

and

$$\begin{aligned} x'''(\theta) = & [\|h\|_{L^1} x'''(\theta) + 2(1 + \|h\|_{L^1} x'(\theta)) \|h\|_{L^1} x''(\theta)] x(\theta) \\ & + [\|h\|_{L^1} x''(\theta) + (1 + \|h\|_{L^1} x'(\theta))^2] (1 + \|h\|_{L^1} x'(\theta)) x(\theta). \end{aligned} \quad (3.19)$$

Note that $x(0) = 1$. By letting $\theta = 0$, we get (2.29). \square

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